Journal of Geometry and Physics 8 (1992) 317–325
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Infinite-dimensional gauge groups and special nonlinear gravitons

R.S. Ward

Department of Mathematics, University of Durham, Durham DH1 3LE, UK

Received 8 July 1991

A gauge theory in flat space–time, in which the gauge algebra is the (infinite-dimensional) algebra of vector fields on a surface, determines a curved space–time metric. This note deals with some completely integrable examples, concentrating on the $N \to \infty$ limit of the Euler–Arnol'd equations [geodesics on SO(N)]. In this case, the metric turns out to be flat, which points the way to a coordinate transformation that solves the original equations.

Keywords: *infinite-dimensional gauge groups, nonlinear gravitons*
1991 MSC: *22E65, 81R10*

1. Introduction

At about the time I arrived in Oxford as a graduate student, Roger Penrose made a particularly important breakthrough: a twistor correspondence for self-dual vacuum space–times which he called the Nonlinear Graviton (Penrose 1976). This opened up a whole new subject, the application of twistor theory to nonlinear systems; and stimulated interest in matters twistorial amongst an even wider audience. This note is a small contribution to the subject.

The idea is as follows. There are several completely integrable differential equations (both ordinary and partial) which involve arbitrary Lie algebras. Indeed, many (perhaps all) of these may be obtained by reduction from the self-dual Yang–Mills equation in four dimensions; and the latter, being a gauge theory, involves an arbitrary Lie algebra (which is inherited by the reduced system). If one now takes the Lie algebra to be an (infinite-dimensional) algebra of vector fields on some manifold M, then the system determines a curved metric. Under the right circumstances, this geometry is four dimensional and self-dual, i.e., we have a nonlinear graviton. Some systems lead to the generic nonlinear graviton (for example, the Nahm equations, with M being three dimensional). For others, however, one gets a restricted class of nonlinear graviton; and this may point the way to a coordinate transformation which solves (or drastically simplifies) the original equation.

The main example studied in this note is that of geodesics on SO(N) equipped with the so-called Manakov metric. This generalizes the "classical" integrable system of Euler's equations for a top with one point fixed (which corresponds to $N = 3$). The limit $N \to \infty$ corresponds (in some sense) to replacing SO(N) by a certain Lie algebra of vector fields on the two-dimensional torus T^2. This "limit" system [geodesics on SO(∞)] determines a self-dual metric, which, however, turns out to be flat; related to this is the fact that the equations can be completely solved (at least implicitly) by means of a coordinate transformation.

Two other examples are mentioned in brief: the SU(∞) Nahm equations and the SU(∞) Toda field equations. Each of these leads to a nonlinear graviton which admits a Killing vector field; but the nature of the Killing vector is different in the two cases. The Nahm equations can be linearized by means of a coordinate transformation, whereas the Toda equation is strictly nonlinear.

The main motivation of all this is towards understanding the relationship between various integrable systems, each of which can be understood in twistorial terms. It presumably contributes little towards the breakthrough in theoretical physics that Roger has always regarded as being the main aim of twistor theory. But such mathematical curiosities and amusements may still count as steps in the right direction.

2. Geodesics on SO(N)

First, let me recall the (well-known) description of geodesics on a Lie group equipped with a left-invariant metric. Take G to be an n-dimensional Lie group, with $\mathcal{G}$ denoting its Lie algebra. An element A^α of $\mathcal{G}$ carries an abstract n-dimensional index α; the Lie product of A^α and B^α is

$$[A,B]^\alpha = C_{\beta\gamma}{}^\alpha A^\beta B^\gamma \tag{1}$$

(i.e., the $C_{\beta\gamma}{}^\alpha$ are the structure constants of $\mathcal{G}$). A left-invariant metric on the group G corresponds to a metric $h_{\alpha\beta}$ on the Lie algebra $\mathcal{G}$. The condition for this left-invariant group metric to be right invariant as well, is that $h_{\alpha\beta}$ be invariant under the adjoint action of G on $\mathcal{G}$; this in turn is equivalent to requiring that

$$C_{\beta\gamma\alpha} := C_{\beta\gamma}{}^\delta h_{\alpha\delta} \tag{2}$$

be totally antisymmetric in $\alpha\beta\gamma$.

If $\Gamma : t \mapsto \Gamma(t) \in G$ is a parametrized curve in G, then

$$A^\alpha := (\Gamma^{-1}\dot{\Gamma})^\alpha \tag{3}$$

is a curve in $\mathcal{G}$ (here an overdot denotes $\mathrm{d}/\mathrm{d}t$). Geodesics on G correspond to extremals of the Lagrangian

$$h_{\alpha\beta}A^{\alpha}A^{\beta}; \tag{4}$$

and this leads to the equations of motion

$$\dot{A}^{\alpha} = -C^{\alpha}{}_{\beta\gamma}A^{\beta}A^{\gamma} \tag{5}$$

(the indices of C are lowered and raised with $h_{\alpha\beta}$ and its inverse). Notice that, if the metric on $\mathcal{G}$ is bi-invariant, so that (2) is skew in $\gamma\alpha$, then the right-hand side of (5) vanishes, and the geodesic equations become a set of first-order ODEs for $\Gamma(t)$. But we are interested here in the more general situation in which the right-hand side of (5) is nonzero.

Suppose now that $\mathcal{G}$ is semisimple, so that it admits an ad-invariant metric $k_{\alpha\beta}$. We can then rewrite (5) in a different form, using the fact that $C_{\beta\gamma}{}^{\delta}k_{\alpha\delta}$ is totally skew. Namely, (5) is equivalent to

$$\dot{B} = [B, A], \tag{6}$$

where

$$B^{\alpha} := k^{\alpha\beta}h_{\beta\gamma}A^{\gamma}. \tag{7}$$

If we think of the inverse of (7) as defining a linear map $L : \mathcal{G} \to \mathcal{G}$, i.e.,

$$L(B)^{\alpha} := h^{\alpha\beta}k_{\beta\gamma}B^{\gamma},$$

then the setup can be restated as follows.

Let $\langle\ ,\ \rangle$ denote an ad-invariant metric on $\mathcal{G}$, and let $L : \mathcal{G} \to \mathcal{G}$ be an invertible linear map. Then

$$h(B,B) := \langle L^{-1}(B), B\rangle \tag{8}$$

defines another metric on $\mathcal{G}$, and the geodesics of the corresponding left-invariant metric on G are given by solutions $t \mapsto B(t) \in \mathcal{G}$ of

$$\dot{B} = [B, L(B)]. \tag{9}$$

If G is SO(3), then these are Euler's equations for the free motion of a rigid body with one point fixed; they provide a classic example of a completely integrable system. For general G, (9) are called the Euler–Arnol'd equations, and they may or may not be integrable (depending on the choice of L). Manakov (1976) demonstrated integrability for $G = \mathrm{SO}(N)$, and L having a particular form (described further below). And if we restrict to linear maps

L which are diagonal (with respect to an appropriate basis for $\mathcal{G}$), then these "Manakov metrics" are the *only* ones for which (9) is completely integrable (Adler and van Moerbeke 1982, Haine 1984).

The Manakov family of linear maps L may be derived as follows. Let B and A be real antisymmetric $N \times N$ matrices, i.e., belonging to the Lie algebra of $\mathrm{SO}(N)$. Let P and Q be constant real diagonal matrices

$$P := \mathrm{diag}(p_1, \ldots, p_N), \qquad Q := \mathrm{diag}(q_1, \ldots, q_N).$$

Impose the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(B + \zeta Q) = [B + \zeta Q, A + \zeta P], \tag{10}$$

for all values of the complex parameter ζ. The coefficient of ζ in (10) gives

$$[A, Q] = [B, P], \tag{11}$$

which determines A in terms of B; in fact, $A = L(B)$, where the entry in the ith row and jth column of $L(B)$ is

$$L(B)_{ij} := \frac{p_i - p_j}{q_i - q_j} B_{ij} \qquad (i \neq j) \tag{12}$$

(no sum over i or j). This, or rather the corresponding left-invariant metric on $\mathrm{SO}(N)$, is the Manakov metric. The remaining content of (10) is just the geodesic equation (9). If $N = 3$, then imposing (12) is effectively no restriction on L; but for $N \geq 4$ it is.

Equation (10) is in Lax form, and is equivalent to saying that the two linear operators

$$B + \zeta Q, \qquad \mathrm{d}/\mathrm{d}t + A + \zeta P \tag{13}$$

commute. One consequence of this is that the spectrum of the matrix $B + \zeta Q$ is conserved, for all ζ. This provides enough constants of motion to ensure that the system is completely integrable in terms of Riemann θ-functions (Manakov 1976). The time evolution of the system amounts to linear flow on an abelian variety, the jacobian of a Riemann surface (which in turn is a branched cover of the ζ-sphere).

How does twistor theory enter into all this? One way is from the fact that (10) is a special case (reduction) of the self-dual Yang–Mills (sdYM) equations, solutions of which correspond to holomorphic vector bundles over (part of) twistor space CP^3 (see, for example, Ward 1987). One can link the twistor geometry to the algebraic geometry referred to in the previous paragraph. But this is not the subject of the present note.

3. Geodesics on SO(∞)

Does there exist, in some sense, an $N \to \infty$ limit of this integrable system? This involves selecting some version of SO(∞), or of the Lie algebra so(∞). There are several such versions (non-isomorphic to one another). The one used here identifies so(∞) as an algebra $\mathcal{F}$ of volume-preserving vector fields on the two-dimensional torus T^2 (Fairlie et al. 1990). Actually, the algebra of all such vector fields is identified as su(∞), and so(∞) is then obtained as a subalgebra. This version of so(∞) is slightly bogus: it is based on taking the $N \to \infty$ limit of the structure constants of so(N) in an appropriate basis, and observing that one then obtains the structure constants of the vector field algebra $\mathcal{F}$. This so(∞) does not appear to contain so(N) in a natural way.

Let x and p, each periodic with period 2π, serve as coordinates on T^2. To a function $f(x,p)$ on T^2 corresponds the volume-preserving (Hamiltonian) vector field

$$\boldsymbol{f} := \frac{\partial f}{\partial x}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial x}.$$

So we identify vector fields as functions (modulo constants). The Lie algebra operation on these functions is just the Poisson bracket. The algebra of all functions is su(∞), and we take $\mathcal{F}$ = so(∞) to be the subalgebra of functions $f(x,p)$ that are odd in p.

So the matrices A, B that appeared previously become vector fields $\boldsymbol{A}$, $\boldsymbol{B}$ corresponding to functions $A(t,x,p)$, $B(t,x,p)$ (both odd in p). The analogues of the diagonal matrices P, Q are functions $P(x)$, $Q(x)$ of x only. Equation (11), with the brackets now being Poisson brackets, implies that $A = L(B)$, where L is simply multiplication by the function

$$\xi(x) := P'(x)/Q'(x)$$

(the prime denotes $\mathrm{d}/\mathrm{d}x$). And the equation of motion (9) then gives

$$B_t = -\xi' B B_p \tag{14}$$

(subscripts denoting partial derivatives).

An ad-invariant metric on $\mathcal{F}$ is

$$\langle f, g\rangle := \int f g \,\mathrm{d}x\,\mathrm{d}p;$$

and so the metric h corresponding to L in this case is

$$h(f,g) = \int \xi^{-1} f g \,\mathrm{d}x\,\mathrm{d}p. \tag{15}$$

Roughly speaking (and ignoring technical difficulties), the group $SO(\infty)$ obtained by exponentiating $\mathcal{F}$ is the group of volume-preserving diffeomorphisms of T^2 which also preserve the involution $p \mapsto -p$. The metric (15) determines a left-invariant metric on this $SO(\infty)$; and the geodesics with respect to this metric correspond to solutions of (14).

As is well known, one can write down the general solution of (14), in implicit form. Namely, if $B(0,x,p)$ is the initial data (at time $t = 0$), put

$$H(x,p) := \xi'(x)B(0,x,p);$$

then $B(t,x,p)$ is determined implicitly by

$$\xi' B(t,x,p) = H(x, p - \xi' t B(t,x,p)). \tag{16}$$

In other words, the geodesic equations on $SO(\infty)$, with left-invariant metric given by (15) for some $\xi(x)$, can be completely solved (in this implicit sense). One way of understanding this is as follows. The two commuting linear operators

$$\boldsymbol{B} + \zeta \boldsymbol{Q}, \quad \partial/\partial t + \boldsymbol{A} + \zeta \boldsymbol{P} \tag{17}$$

are now vector fields; they determine a self-dual vacuum space–time which in this instance is *flat* (details below); this means that a coordinate transformation trivializes the system; the solution (16) in effect expresses this coordinate transformation.

Actually, the vector fields (17) live on a three-dimensional space $R \times T^2$, so as it stands we would get a reduction of the usual nonlinear graviton. Let us avoid this by introducing an extra variable u (which plays no essential role), and replacing (17) by

$$\boldsymbol{B} + \zeta \boldsymbol{Q}, \quad \frac{\partial}{\partial t} + \zeta \frac{\partial}{\partial u} + \boldsymbol{A} + \zeta \boldsymbol{P}, \tag{18}$$

which now are vector fields on $R^2 \times T^2$.

A convenient form of the nonlinear graviton theorem is as follows (Ashtekar et al. 1988; Mason and Newman 1989; Ward 1990c). Let $V_1 + \zeta V_2$ and $V_3 + \zeta V_4$ be two commuting vector fields (each linear in ζ) on a four-manifold M, and suppose that each of the V_a preserves a four-form ω on M. Define a function Λ on M by

$$\Lambda := \omega(V_1, V_2, V_3, V_4).$$

Then the contravariant metric

$$\Lambda^{-1}(V_1 \otimes V_4 + V_4 \otimes V_1 - V_2 \otimes V_3 - V_3 \otimes V_2) \tag{19}$$

is a self-dual vacuum metric. (And in fact, *all* self-dual vacuum metrics arise in this way.)

In our case, $M = \mathrm{R}^2 \times \mathrm{T}^2$ and $\omega = \mathrm{d}t \wedge \mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}p$. The V_a may be read off from (18). The function Λ is easily computed as $\Lambda = -Q' B_p$. The corresponding metric (19) is flat: if we transform from the coordinates (x,p) to (Q,B), then (19) equals

$$\partial_t \otimes \partial_B + \partial_B \otimes \partial_t + \partial_Q \otimes \partial_u + \partial_u \otimes \partial_Q.$$

We see therefore that the $\mathrm{SO}(N)$ geodesic equations (which are completely integrable in the usual sense) become virtually trivial in the limit $N \to \infty$, at least if that limit is understood as described above. The trivialization involves a coordinate transformation which interchanges the dependent and independent variables.

4. The SU(∞) Nahm and Toda equations

The same sort of $N \to \infty$ behaviour occurs in other examples. The two mentioned in this section have already been described elsewhere, and are recalled here for purposes of comparison.

The Nahm equation for Lie algebra $\mathcal{G}$ involves three $\mathcal{G}$-valued functions of t, denoted A^j $(j = 1,2,3)$; the equation they satisfy is

$$\dot{A}^j = \tfrac{1}{2}\epsilon^{jkl}[A^k, A^l].$$

If $\mathcal{G}$ is the Poisson bracket algebra $\mathrm{su}(\infty)$ defined earlier, then the A^j become functions of (t,x,p), satisfying

$$\dot{A}^j = \epsilon^{jkl} A^k_x A^l_p. \qquad (20)$$

This set of equations is the condition that the following two vector fields should commute:

$$(A^1 + \mathrm{i}A^2) + \zeta(\partial/\partial t + \mathrm{i}A^3), \qquad (\partial/\partial t - \mathrm{i}A^3) + \zeta(A^1 - \mathrm{i}A^2). \qquad (21)$$

As before, we can introduce an additional variable u, say by making the replacement

$$A^3 \mapsto A^3 + \partial/\partial u.$$

Then (21) become vector fields on a four-dimensional space, and they determine a self-dual vacuum metric. This time, the metric is not flat. However, it clearly possesses a Killing vector, namely $\partial/\partial u$. This Killing vector turns out to be self-dual; and self-dual vacuum spaces with self-dual Killing vectors correspond to solutions of the three-dimensional Laplace equation (cf. Tod and Ward 1979). This means that the (nonlinear) su(∞) Nahm equation (20) should somehow be equivalent to the (linear) Laplace equation. One can see this equivalence quite directly (Ward 1990a): a coordinate transformation which interchanges the dependent and independent variables (A^1, A^2, A^3) and (t, x, p) converts the one equation into the other.

Let us turn now to the Toda model. The two-dimensional Toda field equations, for affine Lie algebras as well as for finite-dimensional ones, can be obtained as a reduction of the self-dual Yang–Mills equations, and hence can be understood in terms of twistors (Ward 1987). To begin with, take the Lie algebra $\mathcal{G}$ to be simple, of rank r. The Toda field consists of r real-valued functions on R^2; these are denoted $\phi_a(z, \bar{z})$ $(a = 1, 2, \ldots, r)$. The Toda equation is

$$\frac{\partial^2}{\partial z\, \partial \bar{z}} \phi_a + \sum_b K_{ab} \exp \phi_b = 0, \tag{22}$$

where K_{ab} is the Cartan matrix of $\mathcal{G}$. This is a completely integrable system; rather than describe the two commuting linear operators which give rise to it, let us move directly to the case $\mathcal{G} = \mathrm{su}(\infty)$.

For this, we take two vector fields on $\mathrm{R}^2 \times \mathrm{T}^2$ of the form

$$\frac{\partial}{\partial z} + \boldsymbol{f} - \zeta \boldsymbol{e}^+, \qquad \zeta \frac{\partial}{\partial \bar{z}} + \boldsymbol{e}^- + \zeta \boldsymbol{g}, \tag{23}$$

where

$$f = f(z, \bar{z}, x), \qquad g = g(z, \bar{z}, x),$$
$$e^{\pm} = e^{\pm}(z, \bar{z}, x, p) = e(z, \bar{z}, x) \exp(\pm \mathrm{i} p).$$

Note that the dependence of these functions on p is rather special: it means that f and g take values in a Cartan subalgebra of $\mathcal{G}$ (consisting of functions of x only), and e^+ corresponds to a set of simple roots of $\mathcal{G}$. The condition for the two vector fields (23) to commute is

$$\frac{\partial^2}{\partial z\, \partial \bar{z}} \phi + \frac{\partial^2}{\partial x^2} \exp \phi = 0, \tag{24}$$

where $\phi = 2 \log e$. This is the su(∞) Toda field equation. The vector fields (23) determine a self-dual metric on $\mathrm{R}^2 \times \mathrm{T}^2$, which again is not flat, and possesses a Killing vector $\partial / \partial p$. In this case, however, the Killing vector is not self-dual. The quotient of $\mathrm{R}^2 \times \mathrm{T}^2$ by $\partial / \partial p$ is a three-dimensional Einstein–Weyl space, and so solutions of (24) determine a class of Einstein–Weyl metrics (Ward 1990b).

Equation (24), which is completely solvable in principle, cannot be solved or linearized by means of a coordinate transformation (since the quotient

Einstein–Weyl space is not flat, in general). If, however, $\phi(z,\bar{z},x)$ depends on z and $\bar{z}$ only through the combination $t = z + \bar{z}$ (so that $\partial_z - \partial_{\bar{z}}$ is another Killing vector, which this time *is* self-dual), then (24) reduces to the Toda equation

$$\frac{\partial^2}{\partial t^2}\phi + \frac{\partial^2}{\partial x^2}\exp\phi = 0; \tag{25}$$

and (25) can be solved, as before, by a coordinate transformation (Ward 1990b).

5. Remark

The items described here are part of a general framework relating gauge theory (in flat space–time) to curved space–time. Whether such ideas will prove to be really useful is not yet clear. One intriguing suggestion (by Lionel Mason) is that the KP equation (a well-known completely integrable system in 2+1 dimensions) might be obtainable by reduction from the self-dual Einstein equations. To investigate this, and other applications of the framework, requires a better understanding of the $N \to \infty$ limit.

References

Adler, M. and P. van Moerbeke, 1982, The algebraic integrability of geodesic flow on SO(4), Invent. Math. 67, 297–331.

Ashtekar, M., T. Jacobson and L. Smolin, 1988, A new characterization of half-flat solutions to Einstein's equations, Commun. Math. Phys. 115, 631–648.

Fairlie, D.B., P. Fletcher and C. Zachos, 1990, Infinite-dimensional algebras and a trigonometric basis for the classical Lie algebras, J. Math. Phys. 31, 1088–1094.

Haine, L., 1984, The algebraic complete integrability of geodesic flow on SO(N), Commun. Math. Phys. 94, 271–287.

Manakov, S.V., 1976, Note on the integration of Euler's equations of the dynamics of n-dimensional rigid body, Func. Anal. Appl. 10, 93–94.

Mason, L.J. and E.T. Newman, 1989, A connection between the Einstein and Yang–Mills equations, Commun. Math. Phys. 121, 659–668.

Penrose, R., 1976, Nonlinear gravitons and curved twistor theory, Gen. Rel. Grav. 7, 31–52.

Tod, K.P. and R.S. Ward, 1979, Self-dual metrics with self-dual Killing vectors, Proc. R. Soc. London A 368, 411–427.

Ward, R.S., 1987, Multi-dimensional integrable systems, Springer Lecture Notes in Physics 280, 106–116.

Ward, R.S., 1990a, Linearization of the SU(∞) Nahm equations, Phys. Lett. B 234, 81–84.

Ward, R.S., 1990b, Einstein-Weyl spaces and SU(∞) Toda fields, Class. Quantum Grav. 7, L95–L98.

Ward, R.S., 1990c, The SU(∞) chiral model and self-dual vacuum spaces, Class. Quantum Grav. 7, L217–L222.